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Computation of the Forcing Term for Boundary Flow through a Permeable Boundary*

JOHN M. BOWNDS

Energy Division, Oak Ridge National Laboratory,[†]
Oak Ridge, Tennessee 37831

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A standard initial-boundary value problem which commonly appears in the theory of groundwater flow through porous media is converted to a boundary integral equation (BIE). The original problem, which evidently has remained unsolved for three decades, can be solved using this BIE approach if a certain driving term can be determined. This paper describes how this driving term can, itself, be determined explicitly as a solution to a mixed Fredholm Volterra integral equation over an infinite domain. The solution of this equation is computable via an approximating system of ordinary differential equations in the time variable. © 1988 Academic Press, Inc.

I. INTRODUCTION

Recent work by the author [4] has shown that boundary integral equation techniques lead to an analytical expression for the solution of a previously unsolved problem in groundwater hydraulics. The previous paper provides detail in the origin of the particular boundary value prrroblem, and the reader should refer to [4] for the derivation of the integral equation formulation and its formal solution.

One of the principal difficulties encountered in solving the proposed boundary integral equation formulation is that of providing an adequate forcing term for the integral equation. A technique for doing so is described in the earlier paper, however, that procedure amounts to solving a large system of first kind Volterra equations which are characterized by what the author has experienced as a very ill conditioned integral operator. Also, that paper alludes to, but does not answer, a question of the *uniqueness* of the determination of certain coefficients.

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Since the term in question is defined physically as the driving flux for the fluid flow through a *porous*, *permeable* boundary (in particular, the flow of groundwater past the boundary between a pumped aquifer and an adjacent aquitard), the correct determination of this forcing term has important implications on the overall flow regime in both regions separated by the boundary. The purpose of the current paper is to describe how this term can be determined by solving a system of *ordinary differential equations* with homogeneous initial conditions. The approach taken is a combination of methods which the author has applied to much simpler Volterra equations on multiple occasions [3, 5, 6, 11]. It is expected that this observation should be useful in an eventual computer solution *based on analytical solutions* of both the boundary integral equation and the original initial-boundary value problem. It is obvious that these problems can be approached using strictly numerical methods. However, for mathematical modeling purposes, it is useful to have expressions from which *explicit* parameter dependence may be deduced. The author has a "bench" version of a preliminary code which shows that, given the driving term which is developed in this paper, the multiple Fourier expansions, which must be computed to effect a solution of the original problem, converge quickly and are not difficult to compute.

To understand how the quantity of interest here relates to the original boundary value problem, it is necessary to review some facts already established in [4]. These expansions are basically generalizations of Hantush's work and are not related to more recent expansions in [8].

II. A Coupled Initial-Boundary Value Problem for the Hydraulic Head in a Leaky Aquifer–Aquitard Confine

In [12–17] and many other previous documents mentioned in [10], the principal problem settles on the determination of $u_i(r, z, t)$, $i=1, 2$, in the following formulation. Solve

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}u_i(r, z, t)\right)+\kappa_i\frac{\partial^2}{\partial z^2}u_i(r, z, t)=\frac{1}{\nu_i}\frac{\partial}{\partial t}u_i(r, z, t), \tag{2.1}$$

for $0<r$, t, $0<z<Z_1<Z_2$ if $i=1$, and $Z_1<z<Z_2$, if $i=2$, with $u_i(r, z, 0)=0$, $i=1, 2$. Also,

$$\frac{\partial u_1}{\partial z}(r, 0, t)\equiv 0, \tag{2.2}$$

$$u_2(r, Z_2, t)\equiv 0, \tag{2.3}$$

$$\lim_{r\to\infty} u_i(r, z, t)=0,$$

for z in the appropriate domains $\{0 \leqslant z \leqslant Z_1$ if $i=1$, $Z_1 \leqslant z \leqslant Z_2$ if $i=2\}$, and

$$\lim_{r \to 0} r \frac{\partial u_1}{\partial r}(r, z, t) = \begin{cases} -q, & \text{if } z \in [d, l] \subset [0, Z_1], \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Additionally, the coupling of u_1 and u_2 must occur at the boundary $z = Z_1$ as

$$\frac{\partial u_1}{\partial z}(r, Z_1, t) \equiv \lambda \frac{\partial u_2}{\partial z}(r, Z_1, t), \qquad \lambda = k_z^{(2)}/k_z^{(1)}, \tag{2.5}$$

$$u_1(r, Z_1, t) \equiv u_2(r, Z_1, t). \tag{2.6}$$

In the aquifer–aquitard hydraulics application described in [4], u_1 and u_2 represent the "drawdowns" in an aquifer of thickness Z_1 and overlying aquitard of thickness $Z_2 - Z_1$, respectively. Further, if $k_z^{(1)}$ and $k_z^{(2)}$ represent the vertical hydraulic conductivities then the quantities

$$k_z^{(1)} \frac{\partial u_1}{\partial z}(r, z, t) \qquad (0 \leqslant z \leqslant Z_1), \tag{2.7}$$

and

$$k_z^{(2)} \frac{\partial u_2}{\partial z}(r, z, t) \qquad (Z_1 \leqslant z \leqslant Z_2), \tag{2.8}$$

are the vertical velocities, respectively. It is noted that $\kappa_i = k_z^{(i)}/k_r^{(i)}$ ($k_r^{(i)}$ is the radial hydraulic conductivity in layer $i=1$ or 2) and $\lambda = k_z^{(2)}/k_z^{(1)}$, the quantity v_i is the ratio of the hydrological "storage coefficient" to the radial hydraulic conductivity.

This initial-boundary value problem would appear to be somewhat straightforward, except that the matching conditions create a complication which necessitates either an ad hoc change in the differential equations (consistent, to some degree, with the intent of the model; see [10]) or the introduction of certain mathematical complexities which amount to the work in [4, 15]. It is this latter complication which is the subject of this paper.

III. A Boundary Integral Equation for $(\partial u_1/\partial z)(r, Z_1, t)$

After a fairly involved derivation, it is shown in [4, 15] that, letting

$$v_i(r, t) = \frac{\partial u_i}{\partial z}(r, Z_1, t), \qquad i = 1, 2, \tag{3.1}$$

it can be established that

$$\begin{pmatrix} v_1(r, t) \\ v_2(r, t) \end{pmatrix} = \begin{pmatrix} f_1(r, t) \\ f_2(r, t) \end{pmatrix} + \int_0^\infty K(r, \rho, t) \begin{pmatrix} v_1(\tau, t) \\ v_2(\rho, t) \end{pmatrix} d\rho, \tag{3.2}$$

where

$$f_1(r, t) = \sum_{n=0}^{\infty} a_n(t) * L^{-1}\{K_0(w_n^{(1)}(p))\}, \tag{3.3}$$

and

$$f_2(r, t) = \sum_{n=1}^{\infty} C_n(t) * L^{-1}\{K_0(w_n^{(2)}(p))\}. \tag{3.4}$$

In these equations, L^{-1} denotes the inverse Laplace transform with $p \in \mathbb{C}$ the transform variable, K_0 denotes the usual modified Bessel function of order zero, and $\{a_n(t)\}$ denotes a set of "pump" coefficients which, incidently, are identical to those first computed in [12]. The matrix kernel K consists of fairly complicated, albeit rapidly convergent, expansions of inverse transforms, all terms being *known*. The $\{C_n(t)\}$ are unknown coefficients which could be computed by generating a certain sequence of equations by using a sequence of moments from the basic matching requirement $u_1(r, Z_1, t) \equiv u_2(r, Z_1, t)$. It is noted that the lack of an orthogonal expansion precludes a successive coefficient determination as with Fourier coefficients. In [4], the actual method for finding these coefficients is laborious, and, in the author's experience, does not provide a very stable numerical procedure for computation, even though there are specific methods available [7]. As shown in [4], the basic conversion to the integral formulation uses straightforward transform techniques as in [9].

It turns out that both analytically and computationally, it may be more advisable *not* to attempt finding the coefficients $C_n(t)$, but, rather, to seek a separate computational procedure for the entire function $f_2(r, t)$ as a solution to a *mixed integral equation* which may be computed by solving an *initial value problem* using well-known algorithms. Before proceeding, it is noted that the given flux matching condition (2.5) implies that the system (3.2) is easily reducible to a single equation.

IV. An Integral Equation for the Driving Boundary Flux $f_2(r, t)$

As indicated above, the system of integral equations is reducible to a single equation. This fact, together with the additional "drawdown" continuity requirement (2.6), eventually implies that the needed function $f_2(r, t)$ must satisfy an equation of the form

$$f_2(r,t)+\int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\left[k_2(r,\rho,t-\tau)+\int_{\sigma=0}^{\infty}k_2(\sigma,\rho,t-\tau)\,d\sigma\right]f_2(\rho,\tau)\,d\rho\,d\tau$$
$$=f_1(r,t)+\int_{\tau=0}^{t}\int_{\rho=0}^{\infty}\left[k_1(r,\rho,t-\tau)+\int_{\sigma=0}^{\infty}k_1(\sigma,\rho,t-\tau)\,d\sigma\right]$$
$$\times f_1(\rho,\tau)\,d\rho\,d\tau. \tag{4.1}$$

For the sake of continuity between this and previous work, the individual known functions are recalled here as

$$f_1(r,t)=\frac{2q}{Z_1}\sum_{n=0}^{\infty}s_n\,W\left(\frac{r^2}{4v_1t},\frac{\sqrt{\kappa_1}\,n\pi r}{Z_1}\right)\left(\frac{\sin(n\pi l/Z_1)-\sin(n\pi d/Z_1)}{n\pi/Z_1}\right), \tag{4.2}$$

$$k_1(r,\rho,t)=\frac{2\sqrt{v_1\rho/r}}{Z_1}\sum_{n=0}^{\infty}D(\sqrt{v_1},r-\rho,t)\exp\left(-\kappa_1 v_1\left(\frac{n\pi}{Z_1}\right)^2 t\right), \tag{4.3}$$

$$k_2(r,\rho,t)=\frac{2\sqrt{v_2\rho/r}}{Z_2-Z_1}\sum_{n=1}^{\infty}D(\sqrt{v_2},r-\rho,t)\exp\left(-\kappa_2 v_2\left(\frac{(n-1/2)\pi}{Z_2-Z_1}\right)^2\right), \tag{4.4}$$

and

$$D(\alpha,r-\rho,t)=\frac{\alpha}{2\sqrt{\pi t}}\exp\left(\frac{-(r-\rho)^2}{4\alpha^2 t}\right). \tag{4.5}$$

The above integral equation will supply the term which, in [4], is required for the resultant boundary flux due to the existence of a (Darcy) pump as modeled by (2.4), the usual radial pump assumption. The remainder of this paper will be a description of the appropriate generalizations of methods previously used by the author to study scalar Volterra equations. The function W is the well function in [12], and the coefficients s_n are taken as Lanczos summability terms [2].

V. Initial Value Problem

For brevity, the notation will be simplified by letting

$$f(r,t)=f_2(r,t), \tag{5.1}$$

$$k(r,\rho,t)=k_2(r,\rho,t)+\int_0^{\infty}k_2(r,\sigma,t)\,k_2(\sigma,\rho,t)\,d\sigma, \tag{5.2}$$

and

$$\begin{aligned} g(r,t) &= f_1(r,t) \\ &\quad + \int_0^t \int_0^\infty \bigg[k_1(r,\rho,t-\tau) \\ &\quad + \int_0^\infty k_1(r,\sigma,t-\tau)\, k_1(\sigma,\rho,t-\tau)\, d\sigma \bigg] f_1(\rho,\tau)\, d\rho\, d\tau. \end{aligned} \tag{5.3}$$

All of these functions are available in rapidly convergent series form as in Section IV. This allows the integral equation in question to be written as

$$f(r,t) + \int_0^t \int_0^\infty k(r,\rho,t-\tau) f(\rho,\tau)\, d\rho\, d\tau = g(r,t). \tag{5.4}$$

A. *Kernel Approximations*

In [5, 6, 11] the author has demonstrated the use of several decompositions for a function of several variables. There are many possibilities such as multiple Taylor series and so on. In any case, it is assumed here that k may be adequately approximated as

$$k(r,\rho,t) = \sum_{i=1}^{n} A_i(r)\, B_i(\rho)\, \varphi_i(t), \tag{5.5}$$

where A_i, B_i, φ_i are at least continuous on $[0,\infty)$ and, further, φ_i is differentiable with

$$\varphi_i'(t) = \sum_{j=1}^{i-1} \gamma_{ij}\, \varphi_j(t), \tag{5.6}$$

where $\{\gamma_{ij}\}$ are known.

These assumptions may be met by using polynomial approximations from any number of sources such as iterated orthogonal polynomial approximations, interpolation, or least-squares fitting using polynomial basis functions. The point is that for the kernel k_2, as described above, a variety of these approximations are possible.

A substitution of the approximation (5.5) yields the *approximate* equation

$$\begin{aligned} u(r,t) &= g(r,t) - \int_0^t \int_0^\infty \sum_{i=1}^{n} A_i(r)\, B_i(\rho)\, \varphi_i(t-\tau)\, u(\rho,\tau)\, d\rho\, d\tau \\ &= g(r,t) - \sum_{i=1}^{n} A_i(r) \int_0^t \int_0^\infty B_i(\rho)\, \varphi_i(t-\tau)\, u(\rho,\tau)\, d\rho\, d\tau. \end{aligned} \tag{5.7}$$

It is assumed here that the fact that $u(r, t) \to 0$ as $r \to +\infty$, together with an implied assumption of the finiteness of the double integral of the approximating basis functions B_i, φ_i, is in effect so that the calculations at hand are (formally) valid.

Now let

$$U_{ij}(t) = \int_0^t \varphi_i(t-\tau) \int_0^\infty B_j(\rho)\, u(\rho, \tau)\, d\rho\, d\tau. \tag{5.8}$$

This choice is motivated by some previous work of the author and, assuming φ_i exists, implies that

$$\begin{aligned} U'_{ij}(t) &= \int_0^t \int_0^\infty \varphi'_i(t-\tau)\, B_j(\rho)\, u(\rho, \tau)\, d\rho\, d\tau \\ &\quad + \varphi_i(0) \int_0^\infty B_j(\rho)\, u(\rho, t)\, d\rho \\ &= \int_0^t \int_0^\infty \sum_{l=1}^{i-1} \gamma_{il}\, \varphi_l(t-\tau)\, B_j(\rho)\, u(\rho, \tau)\, d\rho\, d\tau \\ &\quad + \varphi_i(0) \int_0^\infty B_j(\rho)\, u(\rho, t)\, d\rho \qquad (5.9) \\ &= \sum_{l=1}^{i-1} \gamma_{il} \int_0^t \int_0^\infty \varphi_l(t-\tau)\, B_j(\rho)\, u(\rho, \tau)\, d\rho\, d\tau \\ &\quad + \varphi_i(0) \int_0^\infty B_j(\rho)\, u(\rho, t)\, d\rho \\ &= \sum_{l=1}^{i-1} \gamma_{il} U_{lj}(t) + \varphi_i(0) \int_0^\infty B_j(\rho) \left[g(\rho, t) - \sum_{\nu=1}^{n} A_\nu(\rho)\, U_{\nu\nu}(t) \right] d\rho. \end{aligned}$$

This is an $n \times n$ system of *ordinary* differential equations with coefficients which consist of the computable quantities γ_{il},

$$\int_0^\infty B_j(\rho)\, g(\rho, t)\, d\rho, \tag{5.10}$$

and

$$\varphi_i(0) \int_0^\infty B_j(\rho)\, A_\nu(\rho)\, d\rho. \tag{5.11}$$

Unfortunately, the integral in (5.10) needs to be computed at any point t where the right-hand side of the ordinary differential equation needs to be evaluated. This would seem to discourage the use of any o.d.e. solver having a high overhead due to excessive function evaluations. On the other hand, for analytical purposes, it is noticed that the matrix differential equation can be written as

$$\begin{aligned} U'(t) &= AU(t) + F(t), \qquad t > 0, \\ U(0) &= 0, \end{aligned} \tag{5.12}$$

where all functions are $n \times n$ matrix valued.

Finally, it is noted that Shampine [18] has shown that the *single* variable case of (5.7), even if the equation were nonlinear, yields a tailor-made algorithm for solution. This fact has yet to be taken advantage of and mechanized to the author's knowledge.

VI. A Formulation for the Solution of the Boundary Integral Equation in Terms of a Matrix Exponential

Any generic system (5.12) has as solution

$$U(t) = \int_0^t \exp[-A \cdot (t-s)]\, F(s)\, ds, \tag{6.1}$$

where $\exp(-A)$ is computable as a *finite* sum via spectral decomposition.

In terms of the original goal of producing an approximate solution of the boundary (flux) integral equation in Section IV, it is now reasonably straightforward to see that the author's original approach in [4], namely, attempting to find all the coefficients in a previous nonorthogonal expansion, may have been somewhat off the mark. The current work can now provide a "closed" form solution for the function $f_2(r, t)$ in that

$$f_2(r, t) = g(r, t) - \sum_{i=1}^{n} A_i(r)\, U_{ii}(t), \tag{6.2}$$

where U_{ii}, $i = 1, 2, \ldots, n$, are the *diagonal* elements of the matrix in (6.1). In essence, (6.2) provides the needed result because $g(r, t)$ is given in (5.3) in terms of the known quantities f_1 and k_1, and diag $U(t)$ is, in view of (6.1), dependent on basic quadratures and the (finite) idempotent expansion of the exponential matrix, a quantity which requires the eigenvalues of A. The final result simply amounts to a compilation of these results.

VII. Relation to Initial-Boundary Value Problem; Remaining Work

In [4], the original integral transforms imply that the solutions $u_1(r, z, t)$ and $u_2(r, z, t)$, for $0 \leqslant z \leqslant Z_1$ and $Z_1 \leqslant z \leqslant Z_2$, respectively, are expressible in terms of the boundary flux terms given by the solutions $v_1(r, t)$ and $v_2(r, t)$ which, in turn, depend on the solution $f_2(r, t)$ of the multiple integral equations described above. Now having a recipe for the *input* quantity f_2, it is possible to write

$$\begin{pmatrix} v_1(r, t) \\ v_2(r, t) \end{pmatrix} = \begin{pmatrix} f_1(r, t) \\ f_2(r, r) \end{pmatrix} + \int_0^\infty R(r, \rho, t) \begin{pmatrix} f_1(\rho, t) \\ f_2(\rho, \tau) \end{pmatrix} d\rho, \tag{7.1}$$

where R is the usual resolvent for K in (3.2).

The result to be shown in form here now relates to the original initial-boundary value problem described in Section II. The point is that, as with the typical boundary integral equation approach, the drawdowns u_1 and u_2 may be expressed as *quadratures* of known quantities. In particular, Eqs. (4.1) and (4.2) and [1] imply that

$$\begin{aligned} u_1(r, z, t) = {} & \frac{2}{Z_1} \sum_{n=0}^{\infty} A_n(t) * L^{-1}\{K_0(w_n r)\} \\ & + \frac{2\lambda}{Z_1} \int_0^\infty \sum_{n=0}^{\infty} L^{-1}\{G(w_n r, w_n \rho)\} * u_2(r, t)\, \rho\, d\rho, \end{aligned} \tag{7.2}$$

for $0 \leqslant Z \leqslant Z_1$, and

$$u_2(r, z, t) = f_2(r, t) + \lambda^{-1} \frac{\pi}{Z_2 - Z_1} \int_0^\infty \sum_{n=1}^{\infty} L^{-1}\{G(w_n r, w_n \rho)\} * u_1(r, t)\, \rho\, d\rho, \tag{7.3}$$

where $\lambda = k_{2z}/k_{1z}$ and G is the Green's function in [4]. Here, as shown previously, the indicated inverse transforms are taken as

$$A_n(t) * L^{-1}\{K_0(w_n r)\} = W\left(\frac{r^2}{4v_1 t}, \frac{\sqrt{\kappa_1}\, n\pi r}{Z_1}\right), \tag{7.4}$$

with $W(\,,\,)$ denoting the Hantush well function [12], and

$$L^{-1}\{G(w_n r, w_n \rho)\} = \frac{v_1}{2\sqrt{r\rho \cdot \pi v_1 t}} \cdot \exp\left\{ -\frac{(r-\rho)^2}{4v_1 t} - \kappa_1 v_1 \left(\frac{n\pi}{Z_1}\right)^2 t \right\}, \tag{7.5}$$

provided t is small.

REFEENCES

1. M. ABRAMOWITZ AND I. STEGUN, "Handook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables," U. S. Dept. of Commerce, Applied Mathematics Series Vol. 55, U. S. Govt. Printing Office, Washington, D. C., 1964.
2. G. ARFKEN, "Mathematical Methods for Physicists," 2nd ed., Academic Press, New York, 1970.
3. C. BAKER AND G. MILLER, "Treatment of Integral Equations by Numerical Methods," Academic Press, New York/London, 1982.
4. J. BOWNDS, A formal solution to a classical initial-boundary value problem in groundwater hydraulics, *Appl. Math. Comput.* **26** (1988), 333–354.
5. J. BOWNDS AND B. WOOD, On numerically solving nonlinear Volterra integral equations with fewer computations, *SIAM J. Numer. Anal.* **13** (1976), 705–719.
6. J. BOWNDS AND B. WOOD, A smoothed projection method for singular nonlinear Volterra equations, *J. Approx. Theory* **25** (1979), 120–141.
7. H. BRUNNER, On the approximate solution of first kind integral equations of the Volterra type, *Computing* **13** (1974), 67–79.
8. Z. CHEN, Z. PANG, L. JIANG, AND M. LIN, Exact solution for the problem of crossflow in a bounded two-aquifer system with an aquitard, *Water Resources Res.* **22**, No. 8 (1986), 1225–1236.
9. R. CHURCHILL, "Operational Mathematics," 3rd ed., McGraw–Hill, New York, 1972.
10. R. FREEZE AND J. CHERRY, "Groundwater," Prentice-Hall, Englewood Cliffs, NJ, 1979.
11. M. GOLDBERG, "Solution Methods for Integral Equations Theory and Applications," Plenum, New York, 1978.
12. M. HANTUSH, Hydraulics of wells, *in* "Advances in Hydroscience" (V. Chow, ed.), Vol. 1, Academic Press, New York, 1964.
13. M. HANTUSH AND C. JACOB, Non-steady radial flow in an infinite leaky aquifer, *Trans. Amer. Geophys. Union* **36** (1955), 95–112.
14. D. LEE, "Internal Communications in Project Status Report," Oak Ridge National Laboratory, Jan. 1985, Oak Ridge, TN.
15. D. LEE AND J. BOWNDS, "Hydrodynamics of Partially Penetrating Wells in Leaky Aquifer System," Technical Letter Report, ORNL/NRC/LTR-86/14, Oak Ridge National Laboratory, May 1986, Oak Ridge, TN.
16. S. NEUMAN AND A. WITHERSPOON, Theory of flow in a confined two aquifer system, *Water Resources Res.* **5** (1969), 803–816.
17. S. NEUMAN AND A. WITHERSPOON, Applicability of current theories of flow in leaky aquifers, *Water Resources Res.* **5** (1969), 817–829.
18. L. SHAMPINE, Personal communication, 1986.